THEORY OF THERMAL INSTABILITY OF THE FLOW OF A VISCOELASTIC FLUID

L. M. Buchatskii, A. M. Stolin, and S. I. Khudyaev

UDC 532.135

The problem of the stability of the flow of viscoelastic fluids has fundamental importance for the technology of the production of polymer products and viscosimetry. This problem is not reduced only to classical inertial turbulence. A number of other mechanisms leading to flow instability are known [1, 2]. A thermal mechanism based on the allowance for dissipative heating and elastic properties within the framework of a linear model of a viscoelastic fluid was drawn upon to explain this phenomenon in [1]. The possibility of a self-oscillatory mode of flow was demonstrated on the basis of a qualitative analysis of the rheological equation and the equation of heat balance in application to simple shear flow and uniform stretching. A theoretical analysis of the self-heating of flowing systems possessing viscoelastic properties is carried out in the present report. The main laws of the thermal instability of viscoelastic fluids discovered in [1] are described.

1. Statement of the Problem

At the foundation of the analysis we place the Maxwell linear model of a viscoelastic fluid,

$$\bar{D} = \frac{1}{G}\frac{d\bar{\tau}}{dt} + \frac{1}{\mu}\bar{\tau}, \tag{1.1}$$

where $\bar{\tau}$ is the stress tensor; $\bar{D}$ is the deformation-rate tensor; G is the elastic modulus; μ is the viscosity; t is the time.

In the general case G and μ are functions of the thermodynamic quantities such as the temperature T and the density ρ. For simplicity in the future we take G = const and ρ = const, although the viscosity is assumed to depend on temperature: $\mu = \mu(T)$. The main assumptions about the character of the flow are reduced to the requirements of:

a) uniformity of the deformation, i.e., constancy of $\bar{D}$ over the volume of the fluid, with the dependence $\bar{D}(t)$ assumed to be known;

b) constancy of the temperature over the volume of the fluid. Under these assumptions Eq. (1.1) is closed by one equation of heat balance

$$c\rho dT/dt = q(\bar{\tau}, T) - \alpha(T - T_0), \tag{1.2}$$

where α is the coefficient of heat transfer from the surface of the fluid; c is the specific heat of the fluid; T_0 is the temperature of the surrounding medium. The dissipative function $q(\bar{\tau}, T)$ is determined by the irreversible viscous deformation, and for simple shear flow it has the form

$$q(\bar{\tau}, T) = |\bar{\tau}|^2/\mu(T) = \sum_{i,j}\tau_{ij}^2/\mu(T).$$

The described model is analyzed for Couette flow between two coaxial cylinders rotating about the axis with different constant velocities. In this case only the tangential components $D_{r,\varphi} = D$ and $\tau_{r,\varphi} = \tau$ of the tensors $\bar{D}$ and $\bar{\tau}$ are different from zero and Eq. (1.1) is scalar. We will take D = const. The temperature dependence of the viscosity is assigned by the Reynolds equation:

$$\mu = \mu_0 \exp[-\beta(T - T_0)] \quad (\mu_0, \beta \text{ — const}).$$

Chernogolovka. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 115-122, May-June, 1979. Original article submitted March 13, 1978.

Three characteristic times can be distinguished in the problem:

$$t_0 = (c\rho)/\alpha,\ t_1 = (c\rho)/(\beta\mu_0 D^2),\ t_2 = \mu_0/G, \tag{1.3}$$

where t_0 is the time of heat transfer; t_1 is the time of heat release; t_2 is the elastic relaxation time.

We introduce the dimensionless quantities through the equations

$$\Theta = \beta(T - T_0),\ \sigma = \tau/(GDt_0),\ x = t/t_0,\ \delta = t_0/t_2,\ \varkappa = t_0/t_1. \tag{1.4}$$

Then Eqs. (1.1) and (1.2) take the form

$$d\sigma/dx = 1 - \delta\sigma e^{\Theta},\ d\Theta/dx = \varkappa\delta^2\sigma^2 e^{\Theta} - \Theta. \tag{1.5}$$

The choice of dimensionless parameters in (1.4) is different from that of [1]. The parameters $\varkappa$ and δ express the intensity of heat release and the variation in elastic energy with respect to the heat-transfer intensity, respectively. An isothermal mode of flow occurs as $\varkappa \to 0$ while an adiabatic mode occurs as $\varkappa \to \infty$. The case of $\delta \to 0$ corresponds to a soft spring in the elastic element of the Maxwell model, when the elastic deformation develops slowly ($t_2 \to \infty$), while the case of $\delta \to \infty$ corresponds to a stiff spring, when the behavior of the model during deformation is determined by the viscous element.

2. Qualitative Analysis

As a function of the structure of the stationary point (σ_0, Θ_0) in the system (1.5), determined from the equations

$$\delta = \frac{1}{\sigma_0} e^{-\Theta_0},\ \varkappa = \frac{\Theta_0}{\sigma_0^2} e^{\Theta_0}, \tag{2.1}$$

the following three regions in the quadrant $\sigma_0 > 0$, $\Theta_0 > 0$ were distinguished in [1]:

$$\Theta_0 > 1 + 1/\sigma_0; \tag{2.2}$$

$$1 + 3/\sigma_0 - 2\sqrt{2(1 + 1/\sigma_0)/\sigma_0} < \Theta_0 < 1 + 1/\sigma_0^{*}; \tag{2.3}$$

$$0 < \Theta_0 < 1 + 3/\sigma_0 - 2\sqrt{2(1 + 1/\sigma_0)/\sigma_0}. \tag{2.4}$$

In the region (2.2), which is the region of instability of the stationary point, the system (1.5) admits a stable limiting cycle. A proof of the existence of a limiting cycle can be found in [3]. Physically this means the existence of a self-oscillatory mode of flow in which the stress and temperature undergo periodic oscillations in time.

In the region (2.3) the stationary point is a stable "focus," which corresponds to a mode of flow with damped oscillations of the stress and temperature in time.

In the region (2.4), where the stationary point is a stable "node," a stable mode of flow is realized.

In the given case the possibility of oscillatory modes is connected with the interaction of an elastic deformation and the thermal factors. On the one hand, an elastic deformation of the spring in the Maxwell model with D = const leads to an increase in the stress, while on the other hand, the dissipative self-heating of the fluid and the decrease in viscosity connected with it cause a decrease in stress. And the concurrence of these factors is the reason for the thermal instability. Here the main point is the strong temperature dependence of the viscosity, which provides for the presence of a "feedback" effect.

We can obtain a description of the regions (2.2)-(2.4) in the plane of the parameters $\varkappa$ and δ. If we set $\xi = 1/\sigma_0$ then from (2.1)-(2.3) we obtain a parametric representation of the boundary Γ_1 separating the region I of self-oscillations from the region II of damped oscillations and of the boundary Γ_2 separating the region II from the region III of stable flow:

$$\Gamma_1: \delta = \xi \exp(-1 - \xi),\ \varkappa = \xi^2(1 + \xi)\exp(1 + \xi);$$

$$\Gamma_2: \delta = \xi \exp(-1 - 3\xi + 2\sqrt{2\xi(1 + \xi)}),\ \varkappa = \xi(1 + 3\xi - 2\sqrt{2\xi(1 + \xi)})/\delta.$$

A diagram of these regions is presented in Fig. 1. From its analysis it is seen that self-oscillations are impossible both at large δ ($\delta > 1/e^2$) and at small $\varkappa$ ($\varkappa < e$). This means that the development of self-oscillations is connected with a certain combination of the viscoelastic properties of the material and of the conditions of heat transfer. In the isothermal case ($\varkappa = 0$) a stable mode of flow occurs for all values of δ. With

*There is a misprint in the expression for the lower boundary of the region (2.3) in [1].

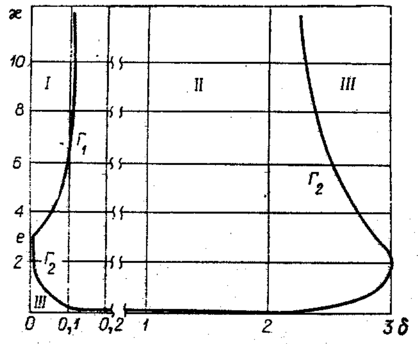

Fig. 1

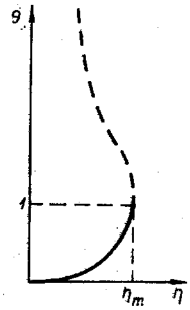

Fig. 2

fixed thermal parameters corresponding to small values of $\varkappa$ ($\varkappa < e$) the oscillatory mode is realized only at intermediate values of the elastic relaxation time t_2, while the flow is stable at sufficiently large and small values of δ. In the adiabatic case ($\varkappa \to \infty$) the oscillation region practically disappears.

3. Limiting Cases

The various limiting relations between the characteristic times (1.3) deserve special consideration. The most interesting prove to be the two limiting cases

$$t_0 \gg t_2(\delta \gg 1),\ t_0 \ll t_2(\delta \ll 1),$$

which we call the cases of mechanical and thermal quasisteadiness, respectively. For their analysis it is convenient to introduce a new normalization of the stress. We set

$$\eta = \delta\sigma = \tau/GDt_2.$$

Then the system (1.5), which we will analyze under zero initial conditions, takes the form

$$\frac{1}{\delta}\frac{d\eta}{dx} = 1 - \eta e^{\Theta},\ \eta(0) = 0,$$
$$\frac{d\Theta}{dx} = \varkappa\eta^2 e^{\Theta} - \Theta,\ \Theta(0) = 0. \tag{3.1}$$

Mechanical Quasisteadiness ($\delta \gg 1$). In this case the elastic deformation is completed long before the end of the process of thermal stabilization ($t_2 \ll t_0$), and then the irreversible viscous component makes the main contribution to the total deformation; i.e., viscous flow occurs. If the flow at $t > t_2$ is considered, then a quasisteady variation in the stress occurs [4], determined only by the thermal factors:

$$\eta = \exp(-\Theta),\ d\Theta/dx = \varkappa e^{-\Theta} - \Theta,\ \Theta(0) = 0.$$

Hence it is seen that there is a single stable mode of flow with steady heating Θ_0 determined by the equality of the heat release and the heat transfer: $\Theta_0 \exp\Theta_0 = \varkappa$. The stress monotonically approaches its limiting value $\eta_0 = \exp(-\Theta_0)$. This approximate solution is valid for all $\varkappa$. In the isothermal case ($\varkappa \to 0$) we have $\Theta_0 \to 0$ and $\eta_0 \to 1$. This case is realized in region III when $\delta > \delta_1 = 3.07$ (Fig. 1).

Thermal Quasisteadiness ($\delta \ll 1$). In this case the slowest process is elastic deformation, and the elastic relaxation time t_2 must be taken as the time scale:

$$x' = \delta x = t/t_2.$$

Then Eqs. (3.1) take the form

$$d\eta/dx' = 1 - \eta e^{\Theta},\ \eta(0) = 0,\ \delta d\Theta/dx' = \varkappa\eta^2 e^{\Theta} - \Theta,\ \Theta(0) = 0.$$

The temperature variation is fine-tuned by the stress variation, and the following quasisteady description [4] is valid:

$$\frac{d\eta}{dx'} = 1 - \eta e^{\Theta},\ \eta(0) = 0; \tag{3.2}$$

$$\left(\frac{1}{\varkappa}\Theta e^{-\Theta}\right)^{1/2} = \eta. \tag{3.3}$$

Equation (3.3) is solvable in the region

$$0 < \eta < \eta_m = 1/\sqrt{\varkappa e}.$$

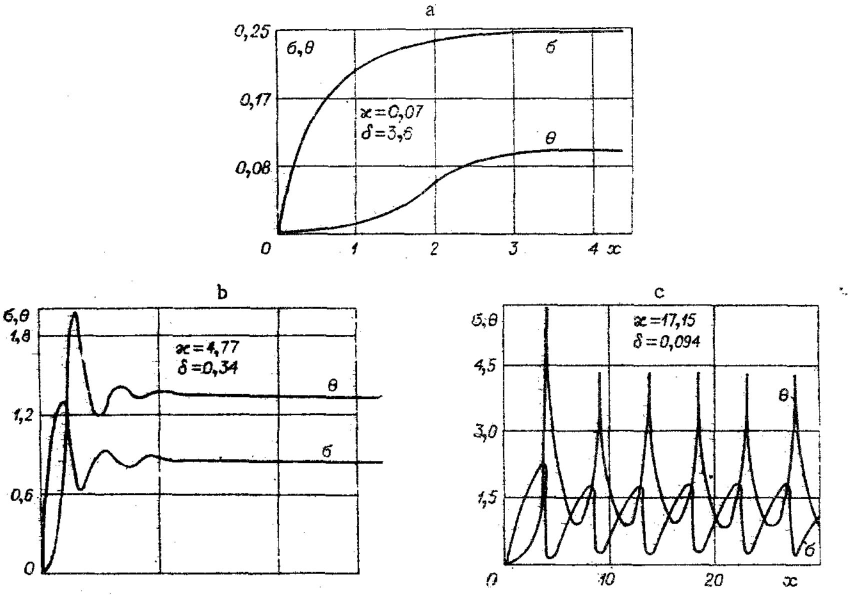

Fig. 3

In this case there are two solutions for the temperature: $\Theta_1(\eta)$ and $\Theta_2(\eta)$ (Fig. 2). According to [4], one must take the lower branch $\Theta_1(\eta)$ of the solution, where $\Theta_1(\eta) < 1$ and $\Theta_1(\eta_m) = 1$.

The time variation in the stress is found from (3.2) using the quadrature

$$x' = \int_0^{\eta} \frac{d\eta}{1 - \eta \exp \Theta_1(\eta)}. \tag{3.4}$$

This solution has qualitatively different natures depending on $\varkappa$.

1. Let $\eta_m e = \sqrt{e/\varkappa} \geqslant 1$. In this case the integral (3.4) diverges at some $\eta = \eta_0 < \eta_m$; i.e., the solution of the system (3.2), (3.3) is definite over the entire time interval $0 < x' < \infty$. The stress and temperature grow monotonically and are bounded: $0 < \eta < \eta_0$ and $0 < \Theta < \Theta_0$.

2. Let $\eta_m e = \sqrt{e/\varkappa} < 1$. In this case the integral (3.4) is definite and bounded in the entire region $0 < \eta < \eta_m$. This means that the stress reaches a finite value $\eta = \eta_m$ in a finite time x_m':

$$x_m' = \int_0^{\eta_m} \frac{d\eta}{1 - \eta \exp \Theta_1(\eta)}. \tag{3.5}$$

Later, when $x > x_m$, a quasisteady solution is absent. The flow has an essentially nonsteady nature. These two types of solutions are separated by the critical condition $\varkappa_* = e$. In Fig. 1 the region of $\varkappa < \varkappa_*$ with $\delta \ll 1$ falls in region III of stable flow while the region of $\varkappa > \varkappa_*$ falls in region I of self-oscillations. Thus, in the given case the absence of a quasisteady solution signifies the emergence into the mode of self-oscillations. The critical condition is written in the following dimensional form:

$$\beta \mu_0 D^2/\alpha = e.$$

This equation shows which parameters affect the formation of self-oscillations.

4. Nonsteady Thermal Modes

A numerical solution of the system (1.5) permits an investigation of the nonsteady properties of the thermal modes of flow. Characteristic curves of the time variation of the stress and temperature in different regions of the parameters $\varkappa$ and δ (see Fig. 1) are presented in Fig. 3a–c. In region III of stable flow the stress and temperature monotonically approach the steady-state values, with the heating–time curves having an inflection (Fig. 3a). In region II the establishment of a steady state is preceded by damped oscillations of the stress and temperature having a sinusoidal nature (Fig. 3b). The frequency ω of the oscillations and the damping decrement Δ are easily expressed through the steady-state values σ_0 and Θ_0 of the stress and temperature:

$$\omega = \frac{1}{2\pi}\sqrt{4\frac{\Theta_0+1}{\sigma_0} - \left(\Theta_0 - \frac{1+\sigma_0}{\sigma_0}\right)^2}, \quad \Delta = \frac{1}{2}\left(\Theta_0 - \frac{1+\sigma_0}{\sigma_0}\right).$$

Equations (2.1) allow one to calculate these quantities through the main parameters $\varkappa$ and δ.

In region I of self-oscillation all the flow characteristics have a relaxation character (Fig. 3c). The amplitude of the self-oscillations increases with greater depth into region I. Sharp thermal self-acceleration and such great heating of the fluid are possible in this case that such nonsteady modes must be considered as explosive. In this case Eq. (3.5) determines the induction period.

Thus, an explosive thermal mode of flow of a viscoelastic fluid proves possible in the presence of a constant deformation rate (D = const). We note that in the flow of a viscous fluid in the model of a rotary viscosimeter [5, 6], modes of hydrodynamic thermal explosion prove to be possible only for a given stress on the moving cylinder. This fundamental difference is connected with the possibility of a progressive increase in heat release in the flow of a viscoelastic fluid under conditions of D = const, which is illustrated by Fig. 4 in which sections of a sharp increase in the heat-release function are seen. For a viscous fluid under these conditions the heat-release function is a decreasing function of time.

An analysis of the (q, Θ) and (σ, Θ) phase diagrams (Fig. 5a, b) is interesting. Three sections can be distinguished rather clearly in the (q, Θ) diagram: a section of progressive heat release, an instantaneous decay due to a discharge of the stress, and a "stagnant zone" where the heat release hardly varies owing to the compensation for a decline in temperature by an increase in stress. The limiting cycle into which the solution of the system (1.5) is "coiled" is shown in the (σ, Θ) diagram.

The most important characteristic of the self-oscillations is the period. On the basis of the analysis and treatment of the numerical results we can offer the empirical equation

$$X = \delta^{-0,71}(2,6\varkappa^{-1/2} + 0.21),$$

which allows one to calculate the oscillation period with an accuracy of 10-15% in the region of $e < \varkappa < 50$ and $0.03 < \delta < e^{-2} = 0.135$.

5. Some Comments

A mathematical model of elastic deformation and of the thermal factors leading to the instability of viscosimetric flow was analyzed above. But the described mechanism of self-oscillations has a general char-

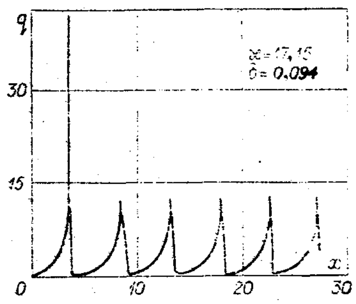

Fig. 4

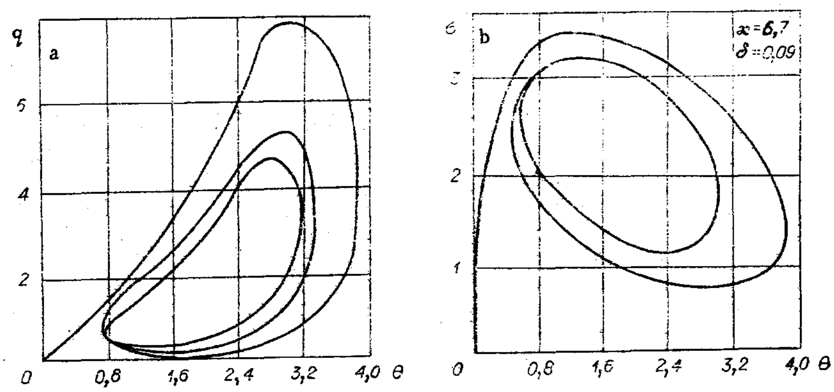

Fig. 5

acter and is interesting in connection with a number of other more complicated phenomena: self-oscillations during the stretching of polymers in the stage of uniform deformation [1] and the formation of a neck [7, 8], deep-focus earthquakes [9], etc. In some cases this mechanism, while not being determining, can be accompanying. This consideration was expressed by S. K. Godunov (in a discussion of [1]) in connection with self-oscillatory phenomena during explosive welding.

The principal factors responsible for thermal instability of the flow of a viscoelastic fluid were taken into account in the model discussed. It allows one to clarify the qualitative aspect of the phenomenon and to make some quantitative estimates. Let us note the role of some details not taken into account in the adopted model. Allowance for the inertia of the dynamometric system has practical importance for a viscosimetric experiment. This leads to a time dependence D(t) of the deformation rate. In this case damped oscillations having a sinusoidal nature are possible even in an isothermal mode during the deformation of a purely viscous fluid. Their superposition onto the self-oscillations of a viscoelastic fluid can only change the quantitative characteristics of the process.

The choice of the exponential temperature dependence of the viscosity adopted in the present work is an unimportant limitation. The appearance of self-oscillations can occur for a wide class of dependences satisfying certain conditions [1]. The allowance for the temperature dependence of the elastic modulus also plays a secondary role.

In connection with the possibility of other mechanisms of instability of the flow of viscoelastic fluids it is interesting to compare the conditions for their occurrence. Thus, in [10] the condition for a transition from a stable mode of flow to an unstable mode of elastic turbulence is connected with the elastic Reynolds number Re_e. The condition for thermal instability found in the present report is connected with the thermal criterion $\varkappa$ of (1.3) and (1.4), which can be represented in the form $\varkappa = (\beta\mu_0/\alpha t_2^2)\,Re_e^2$. Hence it is seen that different combinations of these mechanisms are possible: both their simultaneous action and the action of each of them separately.

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